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A complete asymptotic expansion of power means

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Abstract

The complete asymptotic expansion of power means in terms of Bell polynomials is obtained. Some results recently obtained by M. Bjelica are generalized.

Keywords: Asymptotic expansions; Means

1. Introduction and results

Let $x_i > 0$, $w_i \geqslant 0$, $1 \leqslant i \leqslant n$, $\sum_{i=1}^n w_i = 1$, $\mathbf{x} = (x_1, \dots, x_n)$.

The power mean of order p is defined as

$$M_p(\mathbf{x}) := \left(\sum_{i=1}^n w_i x_i^p\right)^{1/p} \qquad \bigl(p \in \mathbb{R} \setminus \{0\}\bigr) \tag{1}$$

and in the case $p = 0$ as

$$M_0(\mathbf{x}) := \prod_{i=1}^n x_i^{w_i}.$$

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It is easy to show that

$$M_0(\mathbf{x}) = \lim_{p\to 0} M_p(\mathbf{x}).$$

Throughout the paper let $\mathbf{a} = (a, \ldots, a)$ be a point with equal coordinates.

Recently M. Bjelica has studied the asymptotic behaviour of $M_p(\mathbf{x}+\mathbf{a})$ as a tends to infinity. He obtained the following asymptotic relation.

Theorem A. (See [1].) *For each $p \in \mathbb{R} \setminus \{0\}$, the power mean of order p satisfies the asymptotic formula*

$$\begin{aligned} M_p(\mathbf{x}+\mathbf{a}) &= M_1(\mathbf{x}+\mathbf{a}) + (p-1)\big(M_2(\mathbf{x}+\mathbf{a}) - M_1(\mathbf{x}+\mathbf{a})\big) \\ &\quad + o\big(M_2(\mathbf{x}+\mathbf{a}) - M_1(\mathbf{x}+\mathbf{a})\big) \quad (a \to +\infty). \end{aligned}$$

The purpose of this note is to give a complete asymptotic expansion of $M_p(\mathbf{x}+\mathbf{a})$ in the form

$$M_p(\mathbf{x}+\mathbf{a}) = a + \sum_{k=0}^{\infty} \frac{c_k(p)}{a^k} \quad (a \to \infty), \tag{2}$$

where the coefficients $c_k(p)$ are independent of a. We express the coefficients $c_k(p)$ in terms of partial and complete Bell polynomials.

Recall that the (exponential) *partial Bell polynomials* are the polynomials $\mathbf{B}_{m,j}[y_i] := \mathbf{B}_{m,j}(y_1, \ldots, y_{m-j+1})$ in an infinite number of variables $y_1, y_2, \ldots,$ defined by the formal series expansion [2, p. 133, Eq. [3a′]]:

$$\frac{1}{j!}\left(\sum_{k=1}^{\infty} y_k \frac{t^k}{k!}\right)^j = \sum_{m=j}^{\infty} \mathbf{B}_{m,j}[y_i]\frac{t^m}{m!}, \quad j = 1, 2, \ldots. \tag{3}$$

The (exponential) complete Bell polynomials $Y_m[y_i] = Y_m(y_1, y_2, \ldots, y_n)$ (see, e.g., [2, p. 134, Eq. [3b]] are defined by

$$\exp\left(\sum_{j=1}^{\infty} y_j \frac{t^j}{j!}\right) = 1 + \sum_{m=1}^{\infty} Y_m[y_i]\frac{t^m}{m!}, \tag{4}$$

in other words,

$$Y_m[y_i] = \sum_{j=1}^{m} \mathbf{B}_{m,j}[y_i]. \tag{5}$$

Theorem 1. *For each $p \in \mathbb{R}$, the power means $M_p(\mathbf{x}+\mathbf{a})$ possess the complete asymptotic expansion* (2). *In the case $p \neq 0$ the coefficients are given by*

$$c_k(p) = \frac{1}{(k+1)!}\sum_{j=1}^{k+1} j!\binom{1/p}{j}\mathbf{B}_{k+1,j}\left[i!\binom{p}{i}M_i^i(\mathbf{x})\right].$$

In the case $p = 0$ the coefficients are given by

$$c_k(0) = \frac{(-1)^{k+1}}{(k+1)!}\mathbf{Y}_{k+1}\big[-(i-1)!M_i^i(\mathbf{x})\big].$$

Remark 2. Note that the series (2) is not only a complete asymptotic expansion but also a convergent power–series expansion in a^{-1} converging for all $a > \max_{1 \leqslant i \leqslant n} |x_i|$.

The partial Bell polynomials possess the exact expression (see, e.g., [2, p. 134, Theorem A, Eq. [3d]])

$$\mathbf{B}_{m,j}[y_i] = \sum m! \prod_{i=1}^{m-j+1} \left(\frac{1}{k_i!} \left(\frac{y_i}{i!} \right)^{k_i} \right), \tag{6}$$

where the summation takes place over all integers $k_1, k_2, \ldots \geqslant 0$, such that

$$k_1 + 2k_2 + 3k_3 + \cdots = m \quad \text{and} \quad k_1 + k_2 + k_3 + \cdots = j.$$

Taking advantage of this fact we immediately obtain an explicit form of the coefficients $c_k(p)$.

Corollary 3. *For each $p \in \mathbb{R}$, the power means $M_p(\mathbf{x} + \mathbf{a})$ possess the complete asymptotic expansion* (2). *In the case $p \neq 0$ the coefficients are given by*

$$c_k(p) = \sum_{j=1}^{k+1} j! \binom{1/p}{j} \sum \prod_{i=1}^{k-j+2} \left(\frac{1}{k_i!} \left[\binom{p}{i} M_i^i(\mathbf{x}) \right]^{k_i} \right)$$

and the summation in the latter sum runs over all integers $k_1, k_2, \ldots \geqslant 0$, such that

$$k_1 + 2k_2 + 3k_3 + \cdots = k+1 \quad \textit{and} \quad k_1 + k_2 + k_3 + \cdots = j.$$

In the case $p = 0$ the coefficients are given by

$$c_k(0) = (-1)^{k+1} \sum \prod_{i=1}^{k+1} \left(\frac{1}{k_i!} \left[\frac{-1}{i} M_i^i(\mathbf{x}) \right]^{k_i} \right),$$

where the summation runs over all integers $k_1, k_2, \ldots \geqslant 0$ such that

$$k_1 + 2k_2 + 3k_3 + \cdots = k+1.$$

Remark 4. An easy calculation shows that

$$\lim_{p \to 0} c_k(p) = c_k(0) \quad (k = 0, 1, \ldots).$$

For the convenience of the reader we give some explicit expressions of the coefficients $c_k(p)$ which are valid for all $p \in \mathbb{R}$:

$$\begin{aligned}
c_0(p) &= M_1(\mathbf{x}), \\
c_1(p) &= \frac{p-1}{2} \left[M_2^2(\mathbf{x}) - M_1^2(\mathbf{x}) \right], \\
c_2(p) &= \frac{p-1}{3!} \left[(p-2) M_3^3(\mathbf{x}) - 3(p-1) M_1(\mathbf{x}) M_2^2(\mathbf{x}) + (2p-1) M_1^3(\mathbf{x}) \right], \\
c_3(p) &= \frac{p-1}{4!} \big[(p-2)(p-3) M_4^4(\mathbf{x}) - 4(p-1)(p-2) M_1(\mathbf{x}) M_3^3(\mathbf{x}) \\
&\quad - 3(p-1)^2 M_2^4(\mathbf{x}) + 6(2p-1)(p-1) M_1^2(\mathbf{x}) M_2^2(\mathbf{x}) \\
&\quad - (2p-1)(3p-1) M_1^4(\mathbf{x}) \big].
\end{aligned}$$

We remark that the table of partial Bell polynomials in [2, p. 307] may be helpful for the calculation of further coefficients.

As an immediate corollary of Theorem 1 we obtain an extension of the asymptotic linearity (cf. [1, Eq. (1)]), i.e., of the relation

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{M_p(\mathbf{x}) - M_q(\mathbf{x})}{M_r(\mathbf{x}) - M_s(\mathbf{x})} = \frac{p-q}{r-s} \quad (r \neq s).$$

Theorem 5. *Let* $\mathbf{a}$ *be a point with equal coordinates. Furthermore, let* $\mathbf{x}$ *be a point having not all coordinates equal. For all* p, q, r, s *with* $r \neq s$*, we have a complete asymptotic expansion*

$$\frac{M_p(\mathbf{a}+t\mathbf{x}) - M_q(\mathbf{a}+t\mathbf{x})}{M_r(\mathbf{a}+t\mathbf{x}) - M_s(\mathbf{a}+t\mathbf{x})} = \sum_{k=0}^{\infty} q_k(p,q,r,s)\left(\frac{t}{a}\right)^k \quad (t \to 0),$$

where the coefficients $q_k(p,q,r,s)$ *are independent of* t *and* a*. The first two coefficients are given by*

$$q_0(p,q,r,s) = \frac{p-q}{r-s},$$
$$q_1(p,q,r,s) = \frac{1}{3}\frac{p-q}{r-s}(p+q-r-s)\frac{M_3^3(\mathbf{x}) - 3M_1(\mathbf{x})M_2^2(\mathbf{x}) + 2M_1^3(\mathbf{x})}{M_2^2(\mathbf{x}) - M_1^2(\mathbf{x})}.$$

Remark 6. Note that the denominator $M_2^2(\mathbf{x}) - M_1^2(\mathbf{x})$ is different from zero since the power mean $M_p(\mathbf{x})$ is a strictly increasing function of the real variable p unless $\mathbf{x}$ has equal coordinates.

Remark 7. As a special case we obtain an asymptotic relation which can be written in the form

$$\lim_{\mathbf{t}\to\mathbf{0}} \frac{1}{t}\left(\frac{M_p(\mathbf{a}+t\mathbf{x}) - M_q(\mathbf{a}+t\mathbf{x})}{M_r(\mathbf{a}+t\mathbf{x}) - M_s(\mathbf{a}+t\mathbf{x})} - \frac{p-q}{r-s}\right)$$
$$= \frac{p+q-r-s}{3a}\,\frac{p-q}{r-s}\,\frac{M_3^3(\mathbf{x}) - 3M_1(\mathbf{x})M_2^2(\mathbf{x}) + 2M_1^3(\mathbf{x})}{M_2^2(\mathbf{x}) - M_1^2(\mathbf{x})}.$$

Finally, we note that our results are valid for a more general definition of the power mean. Let μ be a positive Borel measure satisfying $\int_{\mathbb{R}} \mathrm{d}\mu(x) = 1$. For each positive and μ-integrable function f we put

$$M_p(f) := \begin{cases} \left(\int_{\mathbb{R}} f^p(x)\,\mathrm{d}\mu(x)\right)^{1/p} & (p \in \mathbb{R}\setminus\{0\}), \\ \exp\left(\int_{\mathbb{R}} \log f(x)\,\mathrm{d}x\right) & (p = 0), \end{cases} \tag{7}$$

provided that the integrals exist (cf. [4, p. 64, Eq. (7)], [3, Eq. (1.2)]). It is obvious that definition (1) is a special case of (7).

The complete asymptotic expansion (2) then reads

$$M_p(f + ae_0) = a + \sum_{k=0}^{\infty} \frac{c_k(p)}{a^k} \quad (a \to \infty),$$

where e_0 denotes the function given by $e_0(x) = 1$ on $\mathbb{R}$.

The proof of the analogue of Theorem 1 is completely similar. Assume that, in addition, f is bounded. Then in the case $p \neq 0$, for $a > 0$, there holds

$$M_p(f+ae_0)=aM_p\bigl(a^{-1}f+e_0\bigr)=a\biggl(\int_{\mathbb{R}}\bigl(1+a^{-1}f(x)\bigr)^p\,\mathrm{d}\mu(x)\biggr)^{1/p}$$

$$=a\Biggl(1+\sum_{k=1}^{\infty}\binom{p}{k}\int_{\mathbb{R}}\bigl(a^{-1}f(x)\bigr)^k\,\mathrm{d}\mu(x)\Biggr)^{1/p}$$

$$=a\Biggl(1+\sum_{k=1}^{\infty}\binom{p}{k}a^{-k}M_k^k(f)\Biggr)^{1/p}$$

$$=a+a\sum_{j=1}^{\infty}\binom{1/p}{j}\Biggl(\sum_{k=1}^{\infty}\binom{p}{k}a^{-k}M_k^k(f)\Biggr)^j.$$

In the case $p=0$, a similar calculation yields

$$M_0(f+ae_0)=aM\bigl(a^{-1}f+e_0\bigr)=a\exp\Biggl(\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{ja^j}M_j^j(f)\Biggr).$$

2. Proofs

Proof of Theorem 1. First we consider the case $p\neq 0$. For $a>\max_{1\leqslant i\leqslant n}|x_i|$, we have

$$M_p(\mathbf{x}+\mathbf{a})=\Biggl(\sum_{i=1}^{n}w_i(x_i+a)^p\Biggr)^{1/p}=a\Biggl(\sum_{i=1}^{n}w_i(1+x_i/a)^p\Biggr)^{1/p}$$

$$=a\Biggl(\sum_{i=1}^{n}w_i\sum_{k=0}^{\infty}\binom{p}{k}\Bigl(\frac{x_i}{a}\Bigr)^k\Biggr)^{1/p}=a\Biggl(\sum_{k=0}^{\infty}\binom{p}{k}\sum_{i=1}^{n}w_i\Bigl(\frac{x_i}{a}\Bigr)^k\Biggr)^{1/p}$$

$$=a\Biggl(\sum_{k=0}^{\infty}\binom{p}{k}a^{-k}M_k^k(\mathbf{x})\Biggr)^{1/p}=a\Biggl(1+\sum_{k=1}^{\infty}\binom{p}{k}a^{-k}M_k^k(\mathbf{x})\Biggr)^{1/p}$$

$$=a+a\sum_{j=1}^{\infty}\binom{1/p}{j}\Biggl(\sum_{k=1}^{\infty}\binom{p}{k}a^{-k}M_k^k(\mathbf{x})\Biggr)^j$$

since $\sup_k M_k(\mathbf{x})\leqslant\max_{1\leqslant i\leqslant n}|x_i|$. From Eq. (3) we deduce

$$\Biggl(\sum_{k=1}^{\infty}\binom{p}{k}\frac{M_k^k(\mathbf{x})}{a^k}\Biggr)^j=j!\sum_{m=j}^{\infty}B_{m,j}\biggl[i!\binom{p}{i}M_i^i(\mathbf{x})\biggr]\frac{a^{-m}}{m!}$$

$(j=1,2,\ldots)$. Therefore, we conclude

$$M_p(\mathbf{x}+\mathbf{a})=a+\sum_{m=1}^{\infty}\frac{1}{m!a^{m-1}}\sum_{j=1}^{m}j!\binom{1/p}{j}B_{m,j}\biggl[i!\binom{p}{i}M_i^i(\mathbf{x})\biggr].$$

In the case $p=0$ we have, for $a>\max_{1\leqslant i\leqslant n}|x_i|$,

$$M_0(\mathbf{x}+\mathbf{a}) = a\prod_{i=1}^{n}\left(1+\frac{x_i}{a}\right)^{w_i} = a\exp\left(\sum_{i=1}^{n} w_i \log(1+x_i/a)\right)$$
$$= a\exp\left(\sum_{i=1}^{n} w_i \sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j}\left(\frac{x_i}{a}\right)^{j}\right)$$
$$= a\exp\left(\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{ja^j}M_j^j(\mathbf{x})\right).$$

With (4) we conclude that

$$M_0(\mathbf{x}+\mathbf{a}) = a + \sum_{m=1}^{\infty}\frac{(-1)^m}{m!a^{m-1}}Y_m\left[-(i-1)!M_i^i(\mathbf{x})\right].$$

This completes the proof of Theorem 1. □

Proof of Theorem 5. With regard to Remark 2 application of Theorem 1 yields the asymptotic formula

$$M_p(\mathbf{a}+t\mathbf{x}) - M_q(\mathbf{a}+t\mathbf{x}) = \frac{t^2}{a}\frac{p-q}{2}\left(M_2^2(\mathbf{x}) - M_1^2(\mathbf{x})\right)$$
$$+ \frac{t^2}{a}\sum_{k=1}^{\infty}\frac{(t/a)^k}{(k+1)!}\sum_{j=1}^{k+1} j!\left[\binom{1/p}{j}\mathbf{B}_{k+1,j}\left[i!\binom{p}{i}M_i^i(\mathbf{x})\right]\right.$$
$$\left. - \binom{1/q}{j}\mathbf{B}_{k+1,j}\left[i!\binom{q}{i}M_i^i(\mathbf{x})\right]\right].$$

Using the analogous relation for $M_r(\mathbf{a}+t\mathbf{x}) - M_s(\mathbf{a}+t\mathbf{x})$ the complete asymptotic expansion follows. The special case

$$M_p(\mathbf{a}+t\mathbf{x}) - M_q(\mathbf{a}+t\mathbf{x})$$
$$= \frac{p-q}{2a}\frac{t^2}{M_2^2(\mathbf{x}) - M_1^2(\mathbf{x})}$$
$$\times\left[1 + \frac{(p+q-3)M_3^3(\mathbf{x}) - 3(p+q-2)M_1(\mathbf{x})M_2^2(\mathbf{x}) + (2p+2q-3)M_1^3(\mathbf{x})}{3a(M_2^2(\mathbf{x}) - M_1^2(\mathbf{x}))}t\right.$$
$$\left. + O\left(t^2\right)\right]$$

as $t \to 0$ leads after some calculations to the explicit expressions of the first two coefficients. □

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